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Universality Class of a Fibonacci Ising Model

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The specific heat of a certain ferromagnetic Fibonacci Ising model is shown to have a logarithmic singularity.

KEY WORDS: Lattice statistical mechanics; Ising models; specific heat; quasiperiodic; Fibonacci sequence.

1. INTRODUCTION

Layered Ising models were first introduced by Fisher[(1)] and then studied in greater detail by several authors.[(2-5)] In particular, for a square lattice the nth-order layered Ising model is defined by the energy of interaction

$$\mathscr{E} = -E_1 \sum_{j=1}^{nM+1} \sum_{k=-N+1}^{N} \sigma_{j,k}\sigma_{j,k+1} - \sum_{j=0}^{M-1} \sum_{l=1}^{n} \sum_{k=-N+1}^{N} E_2(l)\, \sigma_{nj+l,k}\sigma_{nj+l+1,k} \tag{1.1}$$

In the thermodynamic limit $M \to \infty$ and $N \to \infty$, Au-Yang and McCoy[(3)] showed for any finite n and any set $\{E_2(j)\}$ that when $T \to T_c$ the specific heat diverges as

$$c/k = -A(n; \{E_2\}) \ln |1 - T/T_c| + O(1) \tag{1.2}$$

An explicit expression for the amplitude $A(n; \{E_2\})$ was derived by Au-Yang and McCoy.[(3)] They found that this amplitude depends strongly on the arrangement of the energies $\{E_2(j)\}$, in that it can vary from order 1 to exponentially small in n. In the case when $n \to \infty$ and the energies were chosen to be independent random variables, the amplitude $A(n; \{E_2\})$ was

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shown to vanish with probability one. This vanishing of the amplitude in the random limit was interpreted by Au-Yang and McCoy as the absence of a logarithmic singularity in the random Ising model, which was in accord with the earlier studies of McCoy and Wu.[6,7]

Consider now a Fibonacci sequence of symbols defined by

$$S_{n+1} = S_n S_{n-1} \qquad \text{for} \quad n = 1, 2,... \tag{1.3}$$

with $S_0 = B$ and $S_1 = A$ and the product in (1.3) is juxtaposition; for example, $S_2 = AB$, $S_3 = ABA$, $S_4 = ABAAB$, $S_5 = ABAABABA$,..., and the Fibonacci sequence is $S_\infty = \lim_{n \to \infty} S_n$. In the sequence S_n there are F_n symbols, of which F_{n-1} are A's, where F_n are the Fibonacci numbers defined by

$$F_{n+1} = F_{n-1} + F_n \qquad \text{with} \quad F_0 = F_1 = 1 \tag{1.4}$$

In this paper we consider the F_nth-order layered Ising model, where $E_2(j)$ is E_A or E_B, depending upon whether the jth position corresponds to an A or B in the sequence S_n. We prove that the amplitude $A(F_n; \{E_2\})$ converges to a nonzero expression. Furthermore, the limiting expression is quite simple when compared with the general formula for the amplitude $A(F_n; \{E_2\})$. We conclude from this that the specific heat for the Fibonacci Ising model defined by the energy of interaction on a square lattice of size $2N \times M$ is

$$\mathscr{E} = -E_1 \sum_{j=1}^{M} \sum_{k=-N+1}^{N} \sigma_{j,k+1} - \sum_{j=1}^{M-1} \sum_{k=-N+1}^{N} E_2(j)\, \sigma_{j,k} \sigma_{j+1,k} \tag{1.5}$$

where $E_2(j)$ is either E_A or E_B, depending whether j is an A or B in the Fibonacci sequence S_∞, has a logarithmic singularity in the specific heat with amplitude given below [see Eq. (3.7)]. This part is nonrigorous, since it assumes the interchange of $T \to T_c$ with the limit $n \to \infty$ in the F_nth-order layered Ising model.

In Section 2 some preliminary lemmas are proved. In Section 3 the amplitude $A(F_n; \{E_2\})$ is computed in the Fibonacci limit. The amplitude simplifies when $n \to \infty$ because we are able to use Weyl's theorem (see, for example, Hua[8]) on the uniform distribution of the sequence $\{n\tau\}$, $n = 1, 2,...$, where τ is irrational and $\{x\}$ denotes the fractional part of x.

2. SOME LEMMAS

Lemma 2.1. Let $\{x_{k,n} \mid 1 \leqslant k \leqslant n, k \in \mathbb{N}\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $x_{k,n} \to x_k$ uniformly in k as $n \to \infty$.

If the sequence $\{x_k \mid k \in \mathbb{N}\}$ is uniformly distributed on $[0, 1]$, then for any continuous function f defined on $[0, 1]$ we have

$$\frac{1}{n}\sum_{k=1}^{n} f(x_{k,n}) \to \int_0^1 f(x)\,dx \qquad \text{as} \quad n \to \infty$$

Proof. Define

$$I_n = \frac{1}{n}\sum_{k=1}^{n} f(x_k) - \int_0^1 f(x)\,dx$$

Then we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n} f(x_{k,n}) - \int_0^1 f(x)\,dx\right| &\leqslant |I_n| + \left|\frac{1}{n}\sum_{k=1}^{n} [f(x_{k,n}) - f(x_k)]\right| \\ &\leqslant |I_n| + \sup_{1 \leqslant k \leqslant n} |f(x_{k,n}) - f(x_k)|\end{aligned}$$

Now let $\varepsilon > 0$ be given. Since $\{x_k \mid k \in \mathbb{N}\}$ is uniformly distributed, there exists $N_1(\varepsilon)$ such that $n \geqslant N_1(\varepsilon)$ implies $|I_n| < \varepsilon/2$. The function f is uniformly continuous on $[0, 1]$ so there exists $\delta(\varepsilon)$ such that for any $x, x' \in [0, 1]$ satisfying $|x - x'| < \delta(\varepsilon)$, we have $|f(x) - f(x')| < \varepsilon/2$. But $x_{k,n} \to x_k$ uniformly in k as $n \to \infty$, so there exists $N_2(\varepsilon)$ such that $n \geqslant N_2(\varepsilon)$ implies $|x_{k,n} - x_k| < \delta(\varepsilon)$ for all k; and hence, $|f(x_{k,n}) - f(x_k)| < \varepsilon/2$ for all k, thus

$$\sup_{1 \leqslant k \leqslant n} |f(x_{k,n}) - f(x_k)| < \varepsilon/2$$

Choosing $n \geqslant \mathrm{Max}(N_1(\varepsilon), N_2(\varepsilon))$, the lemma is proved. ∎

Lemma 2.2. Let F_n be the Fibonacci numbers defined by (1.4). Then for all $n \in \mathbb{N}$ and all $1 \leqslant k \leqslant F_n - 1$,

$$\left[k\frac{F_{n-1}}{F_n}\right] = [k\alpha]$$

where $\alpha = \lim_{n\to\infty} (F_{n-1}/F_n)$ and $[x]$ is the greatest integer function.

Proof. The fraction F_{n-1}/F_n is the nth continued fraction approximate to α. From the theory of continued fractions[8] we have the basic expression

$$\alpha = \frac{F_{n-1}}{F_n} + \frac{(-1)^n \delta_n}{F_n F_{n+1}}, \qquad 0 < \delta_n < 1 \tag{2.1}$$

From this it immediately follows that for n even

$$[k\alpha] \geqslant \left[k \frac{F_{n-1}}{F_n} \right]$$

and that for n odd

$$[k\alpha] \leqslant \left[k \frac{F_{n-1}}{F_n} \right]$$

We now establish the inequalities in the reverse direction (the above inequalities are true for $1 \leqslant k \leqslant F_n$).

For n even we have, using $k < F_n$,

$$k\alpha < k \frac{F_{n-1}}{F_n} + \frac{\delta_n}{F_{n+1}}$$

so that

$$[k\alpha] \leqslant \left[k \frac{F_{n-1}}{F_n} + \frac{\delta_n}{F_{n+1}} \right]$$

The term on the right-hand side of the above inequality will be $[k(F_{n-1}/F_n)]$ if we can show that

$$\left\{ k \frac{F_{n-1}}{F_n} \right\} + \frac{\delta_n}{F_{n+1}} < 1$$

Let $kF_{n-1} = m'F_n + m$, $0 < m < F_n$, so that $\{k(F_{n-1}/F_n)\} = m/F_n$. Observe

$$\frac{m}{F_n} + \frac{\delta_n}{F_{n+1}} < \frac{m}{F_n} + \frac{1}{F_{n+1}} \leqslant \frac{F_n - 1}{F_n} + \frac{1}{F_{n+1}} = 1 - \left(\frac{1}{F_n} - \frac{1}{F_{n+1}} \right) < 1$$

For n odd we have

$$k \frac{F_{n-1}}{F_n} - \frac{\delta_n}{F_{n+1}} < k\alpha$$

so that

$$\left[k \frac{F_{n-1}}{F_n} - \frac{\delta_n}{F_{n+1}} \right] \leqslant [k\alpha]$$

The term on the left-hand side of the above inequality will be $[k(F_{n-1}/F_n)]$ if we can show that $\{k(F_{n-1}/F_n)\} - \delta_n/F_{n+1} > 0$.

Observe

$$\frac{m}{F_n} - \frac{\delta_n}{F_{n+1}} > \frac{m}{F_n} - \frac{1}{F_{n+1}} \geqslant \frac{1}{F_n} - \frac{1}{F_{n+1}} > 0 \quad \blacksquare$$

Lemma 2.3. The sequence $x_{k,F_n} = \{k(F_{n-1}/F_n)\}$, $1 \leqslant k \leqslant F_n - 1$, converges uniformly in k as $n \to \infty$ to $x_k = \{k\alpha\}$, where $\alpha = \lim_{n \to \infty} (F_{n-1}/F_n)$.

Proof. Using $\{x\} = x - [x]$, we have

$$\begin{aligned} x_{k,F_n} - x_k &= k\left(\frac{F_{n-1}}{F_n} - \alpha\right) - \left[k\,\frac{F_{n-1}}{F_n}\right] + [k\alpha] \\ &= k\left(\frac{F_{n-1}}{F_n} - \alpha\right) \end{aligned}$$

where the second equality follows from Lemma 2.2. From (2.1) we easily see

$$\left|\alpha - \frac{F_{n-1}}{F_n}\right| \leqslant \frac{1}{F_n^2}$$

so we may conclude that

$$|x_{k,F_n} - x_k| \leqslant \frac{1}{F_n}$$

The right-hand side of this last inequality is independent of k and goes to zero as $n \to \infty$. $\blacksquare$

Using the above lemmas, we can now prove the following result:

Proposition 2.4. Let $x_{k,F_n} = \{k(F_{n-1}/F_n)\}$, $1 \leqslant k \leqslant F_n$, and let f be a continuous function defined on $[0, 1]$; then

$$\lim_{n \to \infty} \frac{1}{F_n} \sum_{k=1}^{F_n} f(x_{k,F_n}) = \int_0^1 f(x)\, dx$$

Our final lemma follows:

Lemma 2.5. Let p and q be positive, relatively prime integers, $p > q$. Define

$$S_{p,q} = \left\{\left[n\,\frac{p}{q}\right] \,\middle|\, n \in \mathbb{N}\right\}$$

where $[x]$ is the greatest integer function. Then the number of elements in the set

$$S_{p,q} \cap \{l, l+1,..., l+m\}, \qquad 1 \leqslant l \leqslant p, \quad 0 \leqslant m \leqslant p-1$$

is either $[(m+1)\, q/p]$ or $[(m+1)\, q/p]+1$.

Proof. Write $p = sq + r$, $1 \leqslant r \leqslant q-1$, $s \in \mathbb{N}$, and let $a_k = [k(p/q)]$. Now

$$a_{k+1} - a_k = s + \left[(k+1)\frac{r}{q}\right] - \left[k\frac{r}{q}\right]$$

so that $a_{k+1} - a_k = s$ or $s+1$. For $m+1 < s$,

$$\left[(m+1)\frac{q}{p}\right] < \left[s\frac{q}{p}\right] = \left[1 - \frac{r}{p}\right] = 0$$

This shows that the lemma is true for $m+1 < s$. Now let $a_n, a_{n+1},..., a_{n+N}$ be elements of $S_{p,q}$ that lie in $\{l,..., l+m\}$ such that a_{n-1} and a_{n+N+1} are not in $\{l,..., l+m\}$. Then

$$\begin{aligned} m \geqslant a_{n+N} - a_n &= \left[(n+N)\left(s + \frac{r}{q}\right)\right] - \left[n\left(s + \frac{r}{q}\right)\right] \\ &= Ns + \left[(N+n)\frac{r}{q}\right] - \left[n\frac{r}{q}\right] \\ &\geqslant Ns + \left[N\frac{r}{q}\right] = Ns + N\frac{r}{q} - \left\{N\frac{r}{q}\right\} \end{aligned}$$

where we used $[x+y] \geqslant [x] + [y]$. Thus,

$$N\frac{p}{q} \leqslant m + \left\{N\frac{r}{q}\right\} \leqslant m+1$$

or

$$N \leqslant (m+1)\frac{q}{p}$$

Since N is an integer, $N \leqslant [(m+1)\, q/p]$. Therefore, the number of elements in $S_{p,q} \cap \{l,..., l+m\}$, $N+1$, is less than or equal to $[(m+1)\, q/p]+1$.

Now N is maximal, which means $a_{n+N+1} - a_n > m$. Since $a_{n+N+1} - a_n$ is an integer, we must have $a_{n+N+1} - a_n \geqslant m+1$. Thus,

$$m+1 \leqslant (N+1)\,s + \left[(N+n+1)\frac{r}{q}\right] - \left[n\frac{r}{q}\right]$$
$$= (N+1)\,s + (N+1)\frac{r}{q} - \left\{(N+n+1)\frac{r}{q}\right\} + \left\{n\frac{r}{q}\right\}$$

Hence

$$(N+1)\frac{p}{q} \geqslant m+1 + \left\{(N+n+1)\frac{r}{q}\right\} - \left\{n\frac{r}{q}\right\}$$

or

$$N+1 \geqslant (m+1)\frac{q}{p} + \frac{q}{p}\left(\left\{(N+n+1)\frac{r}{q}\right\} - \left\{n\frac{r}{q}\right\}\right)$$

The second term on the right-hand side is strictly larger than -1,

$$(N+1) > (m+1)\frac{q}{p} - 1 \geqslant \left[(m+1)\frac{q}{p}\right] - 1$$

Since $N+1$ is an integer,

$$N+1 \geqslant \left[(m+1)\frac{q}{p}\right]$$

This proves the lemma. ∎

3. AMPLITUDE $A(F_n; \{E_2\})$ IN THE FIBONACCI LIMIT

The critical temperature T_c for a nth-order layered Ising model satisfies the condition[1,3]

$$\left(\frac{1-z_{1c}}{1+z_{1c}}\right)^{2n} = \prod_{l=1}^{n} z_{2c}^2(l) \tag{3.1}$$

For the Fibonacci layering S_n, (3.1) reduces to

$$\left(\frac{1-z_{1c}}{1+z_{1c}}\right)^{2F_n} = z_{Ac}^{2F_{n-1}} z_{Bc}^{2F_{n-2}} \tag{3.2}$$

As $n \to \infty$, the relation (3.2) becomes

$$\left(\frac{1-z_{1c}}{1+z_{1c}}\right)^2 = z_{Ac}^{2\alpha} z_{Bc}^{2\alpha^2} \tag{3.3}$$

where $\alpha^{-1} = (1+\sqrt{5})/2$ is the golden mean. The amplitude $A(F_n; \{E_2\})$ is given by[3]

$$A(F_n; \{E_2\}) = \beta_c^2 (4\pi F_n)^{-1} (z_{1c}^{-1} - z_{1c}) \times \left(2F_n E_1 + \sum_{l=1}^{F_n} E_2(l)\{z_{2c}^{-1}(l) - z_{2c}(l)\}\right)^2 \Big/ B(F_n; \{E_2\}) \tag{3.4}$$

with

$$B^2(F_n; \{E_2\}) = \sum_{m=0}^{F_n-1} \sum_{l=1}^{F_n} \prod_{j=0}^{m} \left(\frac{1+z_{1c}}{1-z_{1c}}\right)^2 z_{2c}^2(l+j) \tag{3.5}$$

where $z_1 = \tanh(E_1/kT)$, $z_2(j) = \tanh\{E_2(j)/kT\}$. We write $z_A = \tanh(E_A/kT)$ and $z_B = \tanh(E_B/kT)$, and the z's have a subscript c when T is T_c. We can now prove the following:

Proposition 3.1. For the Fibonacci layering S_n we have

$$\lim_{n\to\infty} \frac{B^2(F_n; \{E_2\})}{F_n^2} = \left(\frac{1-x^2}{x^2 \ln x^2}\right)^2 \tag{3.6}$$

where $x = z_{Bc}/z_{Ac}$.

Proof. Using the T_c condition (3.2), we can write

$$\left(\frac{1+z_{1c}}{1-z_{1c}}\right)^{2m+2} \prod_{j=0}^{m} z_{2c}^2(l+j) = x^{2(m+1)F_{n-1}/F_n - 2N_A}, \quad x = \frac{z_{Bc}}{z_{Ac}}$$

where $N_A = N_A(l, m, n)$ is the number of A's in the Fibonacci layering S_n between positions l and $l+m$ (including l and $l+m$), $1 \leqslant l \leqslant F_n$, $0 \leqslant m \leqslant F_n - 1$. For odd n the positions of A's in the Fibonacci layering is generated by $\{[k(F_n/F_{n-1})] \mid k \in \mathbb{N}\}$ and this is essentially the case for even n (the last two elements in any block are interchanged). In either case we may apply Lemma 2.5 to conclude that $N_A(l, m, n)$ is either $[\bar{N}_A]$ or $[\bar{N}_A]+1$, where we define $\bar{N}_A = (m+1) F_{n-1}/F_n$. Thus, we must compute the sum

$$I_n = \frac{1}{F_n^2} \sum_{m=0}^{F_n-1} x^{2\bar{N}_A} \sum_{l=1}^{F_n} x^{-2N_A(l,m,n)}$$

Because of the periodicity we have

$$\sum_{l=1}^{F_n} N_A(l, m, n) = (m+1)\, F_{n-1}$$

or

$$\frac{1}{F_n} \sum_{l=1}^{F_n} N_A(l, m, n) = \bar{N}_A$$

Using this last expression and the fact that $N_A(l, m, n)$ assumes only the values $[\bar{N}_A]$ and $[\bar{N}_A]+1$, we may calculate the number of times $N_A(l, m, n)$ assumes the value $[\bar{N}_A]$ as l ranges from 1 to F_n. A simple calculation shows that this number is $F_n(1 - \{\bar{N}_A\})$, where $\{x\}$ is the fractional part of x.

Thus, we have

$$\begin{aligned} I_n &= \frac{1}{F_n^2} \sum_{m=0}^{F_n - 1} x^{2\bar{N}_A}(F_n(1 - \{\bar{N}_A\})\, x^{-2[\bar{N}_A]} + F_n\{\bar{N}_A\}\, x^{-2[\bar{N}_A]-2}) \\ &= \frac{1}{F_n} \sum_{m=0}^{F_n - 1} x^{2\{\bar{N}_A\}}(1 - \{\bar{N}_A\} + x^{-2}\{\bar{N}_A\}) \end{aligned}$$

Applying Proposition 2.4, we obtain

$$I_n \to \int_0^1 x^{2t}(1 - t + x^{-2}t)\, dt = \left(\frac{1 - x^2}{x^2 \ln x^2}\right)^2$$

as $n \to \infty$. ▮

Using Proposition 3.1 and (3.4), we easily establish Theorem 3.2.

Theorem 3.2. For the Fibonnaci S_n layered Ising model, the amplitude $A(F_n; \{E_2\})$ approaches

$$\begin{aligned} &\frac{\beta_c^2}{4\pi}(z_{1c}^{-1} - z_{1c}) \left|\frac{x^2 \ln x^2}{1 - x^2}\right| \{2E_1 + E_A \alpha(z_{Ac}^{-1} - z_{Ac}) \\ &\qquad + E_B \alpha^2(z_{Bc}^{-1} - z_{Bc})\}^2 \end{aligned} \tag{3.7}$$

as $n \to \infty$, where $x = z_{Bc}/z_{Ac}$, $\alpha^{-1} = (1 + \sqrt{5})/2$, and z_{1c}, z_{Ac}, and z_{Bc} satisfy the T_c condition (3.3).

Remarks

1. Numerically the convergence of $A(F_n; \{E_2\})$ to (3.7) is quite rapid.

2. For $E_A = E_B = E_2$, (3.3) reduces to the Onsager T_c condition and (3.7) reduces to the Onsager amplitude for the divergence of the specific heat.
3. The amplitude (3.7) naturally factors into two pieces. The piece coming from Proposition 3.1 does not depend upon the golden mean α^{-1}, whereas the second piece clearly does.

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